Improved Error Probability Bounds for Block Codes on the Gaussian Channel

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We derive remarkably tight upper and lower bounds on the soft decoding error probability for block codes of length n with M equally likely, equal-energy codewords $\mathbf{c}_i$, $i = 0, \ldots, M-1$, represented in n-dimensional Euclidean space and received in the presence of additive white Gaussian noise. These bounds show significant improvement over well-known bounds, e.g., the union upper bound, Berlekamp's tangential union bound, and Shannon's sphere packing bounds.

Following Shannon [1], we consider the differentially thin conical shell $d\mathcal{S}_n(\theta)$ between two circular cones of half-angles θ and $\theta + d\theta$, each with vertex at the origin and axis passing through the correct codeword $\mathbf{c}_0$. This shell contains a fraction $d\Omega_n(\theta) = \frac{(n-1)\Gamma(n/2+1)}{n\Gamma(\frac{n+1}{2})\sqrt{\pi}}(\sin\theta)^{n-2}d\theta$ of the total solid angle in n-dimensional space. For a given code, a certain fraction of the shell's solid angle falls outside $\mathbf{c}_0$'s Voronoi region. We call this fraction the *code geometry function* $F(\theta)$. It depends on the geometry of the code, but not on the Gaussian noise distribution.

Since the noise is spherically symmetric and the Voronoi region is bounded by hyperplanes through the origin, the probability of codeword error P_e can be written as $P_e = \int_0^\pi F(\theta)\, dP(\theta)$, where $dP(\theta)$ is the probability that the received word falls within $d\mathcal{S}_n(\theta)$, given by $dP(\theta) = \frac{(n-1)d\theta}{2^{n/2}\sqrt{\pi}\,\Gamma(\frac{n+1}{2})}(\sin\theta)^{n-2}\int_0^\infty r^{n-1}e^{-(r^2+n\rho-2r\sqrt{n\rho}\cos\theta)/2}\,dr$, and ρ is proportional to the signal-to-noise ratio (SNR) [1].

This separation of the code geometry and the Gaussian noise distribution has allowed us to develop extremely tight bounds on P_e by first bounding the code geometry function $F(\theta)$. For instance, $F(\theta)$ can be upper bounded by a union-bound summation,

$$F(\theta) \le F_1(\theta) \stackrel{\Delta}{=} \min\left[1, \sum_{i\neq 0}\int_0^{\beta_i(\theta)} d\Omega_{n-1}(\theta_1)\right],$$

where $\beta_i(\theta) = \cos^{-1}(\max[-1, \min[1, \gamma_i\cot\theta]])$, and γ_i is the tangent of half the angle between codewords $\mathbf{c}_0$ and $\mathbf{c}_i$. The ith term in the summation represents the fraction of $d\mathcal{S}_n(\theta)$ that is nearer to $\mathbf{c}_i$ than to $\mathbf{c}_0$.

Similarly, $F(\theta)$ can be lower bounded by union-bounding the fraction of $d\mathcal{S}_n(\theta)$ that is nearer to $\mathbf{c}_i$ than to $\mathbf{c}_0$ but is not within $\mathbf{c}_j$'s Voronoi region, i.e., $F(\theta) \ge F_2(\theta)$, where

$$F_2(\theta) \stackrel{\Delta}{=} \sum_{i\neq 0}\int_0^{\beta_i(\theta)} d\Omega_{n-1}(\theta_1)\left(1 - \min\left[1, \sum_{j\neq 0,i}\int_0^{\beta_{ij}(\theta_1)} d\Omega_{n-2}(\theta_2)\right]\right),$$

where $\beta_{ij}(\theta_1) = \cos^{-1}(\max[-1, \min[1, \gamma_{ij}\cot\theta_1]])$ and γ_{ij} is the tangent of half the angle between the projections of $\mathbf{c}_i$ and $\mathbf{c}_j$ into the subspace orthogonal to $\mathbf{c}_0$. The bounds on $F(\theta)$ can be extended into an alternating series of successively tighter upper and lower bounds, terminating in an exact expression after $(n-1)$ iterations.

Bounds on the code geometry function $F(\theta)$ translate directly into bounds on P_e,

$$\int_0^\pi F_2(\theta)\, dP(\theta) \stackrel{\Delta}{=} P_2 \le P_e \le P_1 \stackrel{\Delta}{=} \int_0^\pi F_1(\theta)\, dP(\theta).$$

The figure illustrates the precision of the new bounds applied to the (8,4) extended Hamming code, for which the exact P_e is known. The improved bounds P_2 and P_4 are virtually indistinguishable from the true P_e, as compared to the standard union bound, Berlekamp's tangential union bound, and Shannon's sphere packing lower bound. We have computed these new bounds for codes up to $n = 72$ and verified that they are extremely tight even at low SNR.

Bounds even tighter than P_1 and P_2 are obtained using $F_1(\theta)$ and $F_2(\theta)$ simultaneously, if the total solid angle of $\mathbf{c}_0$'s Voronoi region is known, e.g., $\int_0^\pi F(\theta)\, d\Omega_n(\theta) = 1 - \frac{1}{M}$ for distance invariant codes. This technique generalizes a procedure developed by Shannon [1] that relies on the monotonicity of the Gaussian probability density. Then

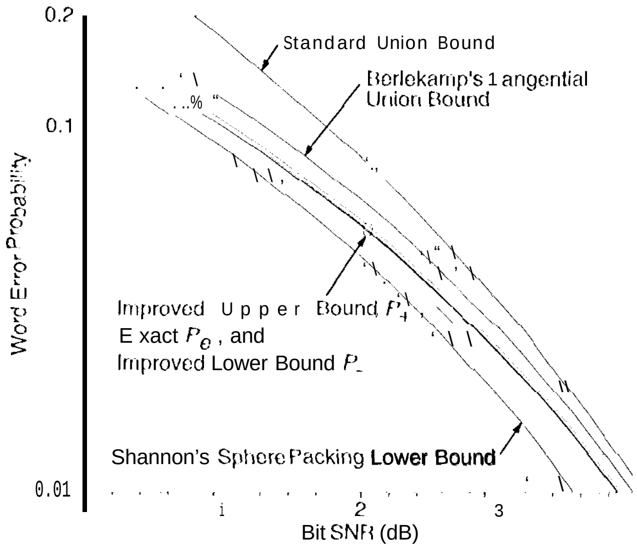
$$P_2 \le \int_0^\pi F_{-1}(\theta)\, dP(\theta) \stackrel{\Delta}{=} P_{-1} \le P_e \le P_{-2} \stackrel{\Delta}{=} \int_0^\pi F_{-2}(\theta)\, dP(\theta) \le P_1$$

where $F_{-1}(\theta) \stackrel{\Delta}{=} \begin{cases} F_2(\theta), \theta \le \theta_{-1} \\ F_1(\theta), \theta > \theta_{-1} \end{cases}$, $\quad F_{-2}(\theta) \stackrel{\Delta}{=} \begin{cases} F_1(\theta), \theta \le \theta_{-2} \\ F_2(\theta), \theta > \theta_{-2} \end{cases}$,

and θ_{-1} and θ_{-2} are any angles large enough to satisfy $\int_0^\pi F_{-1}(\theta)\, d\Omega_n(\theta) \le \int_0^\pi F(\theta)\, d\Omega_n(\theta) \le \int_0^\pi F_{-2}(\theta)\, d\Omega_n(\theta)$.

The technique of separating the code geometry function $F(\theta)$ and the Gaussian noise distribution also simplifies Monte Carlo simulations of code performance. A single simulation may be used to estimate $F(\theta)$ for the code, and then the entire P_e vs. SNR performance curve can be determined by integrating with respect to $dP(\theta)$. The code geometry function $F(\theta)$ may be simulated by generating Gaussian noise samples added to $\mathbf{c}_0$ and performing a decoding operation to determine the fraction of samples at angle θ that fall outside $\mathbf{c}_0$'s Voronoi region. An alternative technique uses $(n-1)$-dimensional Gaussian noise samples orthogonal to $\mathbf{c}_0$ and computes the minimum angle θ at which a scaled version of the noise would cause the received word to leave $\mathbf{c}_0$'s Voronoi region. This latter technique allows each sample to contribute information about $F(\theta)$ for all values of θ.

Other results include exact error probabilities for soft bounded distance decoders, random coding bounds based on an "average" $F(\theta)$, and possible extensions to convolutional codes and lattices.



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References

[1] C.E. Shannon, "Probability of Error for Optimal Codes in a Gaussian Channel", *Bell Syst. Tech. J.*, Vol. 38, pp. 611-656, 1959.